Bifurcations and chaos in a DC-DC buck converter
(Bifurkacje i chaos w układzie obniżającym napięcie)

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The dynamics of DC-DC buck system is studied. System of this type has a broad range of application in power control. There are a lot of cases where electrical energy is processed by power electronics before its final consumption. Power electronic technology is increasingly able to be found in home and workplace. To obtain more completely description of power electronic systems such phenomena as bifurcations characteristic for nonlinear dynamics and chaos are studied [2].

The presence of switching elements, nonlinear components and control methods implies that circuits are nonlinear, time varying dynamical systems. Power converters show strange and unable to observe basis of linear analysis methods behaviour such as bifurcations and chaos. In chaotic operating mode it is possible to observe an undesirable increase in switching loss, and in extreme case switch failure.

Studied system description

DC-DC buck power converter is presented as an example. The subject is one of the simplest but very useful power converters, a DC-DC buck converter, a circuit that converts a direct current (DC) input to a DC output. Many switched mode power supplies employ circuits closely related to it. The experimental example is a second order DC-DC buck converter which output voltage is controlled by a pulse width modulation (PWM) with a constant frequency, working in continuous conduction mode (CCM). The switches in mathematical description are assumed to be ideal. In practise it is necessary to regulate low-pass filter output voltage v against changes in a input voltage and a load current, by adding a feedback control loop as in *Fig. 1* [7].
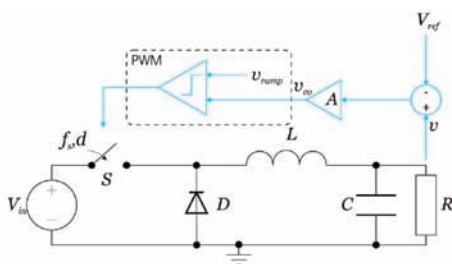

Fig. 1. DC-DC buck converter with feedback control loop
Rys. 1. Schemat badanego układu obniżającego napięcie

The switched operating mode of converters implies a multitopological model in which one particular circuit topology describes the system for a particular interval of time. For constant frequency PWM the operation is cyclic, implying that the topologies repeat themselves periodically. Thus, a natural way to model such kind of operation is to split the system into several subsystems, responsible for describing the system in one subinterval [4].

The discontinuous conduction mode does not take place in considered buck converter, and can be represented by a piecewise linear vector field. Using the notation $x=[v,i]^{\mathrm{T}}$, (y^{T} donates the transpose of y) system description looks as follow.

$$\frac{d\,x}{dt}=f\left(x,t\right) \tag{1}$$

$$f\left(x,t\right)=\begin{bmatrix}-\dfrac{1}{RC} & \dfrac{1}{C}\\[2mm] -\dfrac{1}{L} & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\[2mm]\dfrac{V_{in}}{L}\end{bmatrix}\mathbf{1}_s(t) \tag{2}$$

where:

$$\mathbf{1}_s(t)=0\;if\;t\notin s\;\cup\;\mathbf{1}_s(t)=1\;if\;t\in s \tag{3}$$

also where v is the capacitor voltage and i is the inductor current and the ramp voltage is given by

$$s=\{t\geqslant 0:v_{co}<v_{ramp}(t)\}$$
$$v_{ramp}=V_l+(V_u-V_l)t/T$$

where: V_l and V_u are respectively the lower and upper voltages of the ramp and T its period and

$$v_{co}(t)=A\left(v(t)-V_{ref}\right) \tag{4}$$

where: A is the linear amplifier gain and V_{ref} the reference voltage.

Methods of theoretical model of the buck converter analysis

The nonlinear phenomena include bifurcations (sudden changes in system operation), coexisting attractors (alternative stable operating modes), and chaos. If power converter is going to be designed, a knowledge about these issues existence and its investigation methods is desired. There should be emphasised that linear methods applied alone cannot give a wide spectrum of information of nonlinear phenomena and are insufficient in predicting and system analysing.

Poincaré map

When state vector evolution is known, there is a possibility to discretise it using mapping. Poincaré map is the most widely used discrete time model for DC-DC converters [3,4]. This map can be obtained by sampling the system solution every T seconds, at the beginning of each ramp cycle. This nonlinear method of analysis gives a lot of information about sys-

tem. In contrast to other types of mapping, Poincaré map is able to show aside from bifurcations and chaos also difference between quasiperiodic attractor and strange attractor (chaotic operation).
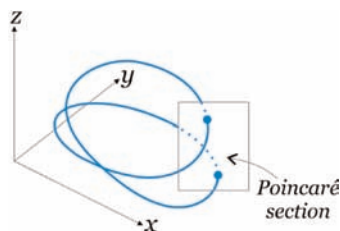

Fig. 2. Poincaré map of limit cycle in the Poincaré plane
Rys. 2. Prezentacja odwzorowania Poincaré cyklu granicznego na płaszczyźnie Poincaré

Bifurcations

In the study of dynamical systems the appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation. Thus, a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation value [3]. If one varies bifurcation parameters the phase portrait may deform slightly without altering its qualitative (i.e., topological) features, or sometimes the dynamics may be modified significantly, producing a qualitative change in the phase portrait.

If an iterative map is used to model the system, the linearised system needs to be examined. Suppose the iterative map is:

$$x \rightarrow f(x, \alpha) \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^1,$$

then the Jacobian $J_f = \partial f/\partial x^T$ characterising the linearised system is given by evaluated at the fixed point. The eigenvalues of system can be obtained by solving the characteristic equation $\det(\mu \mathbf{1} - J_f) = 0$.

If one of the eigenvalues is observed to move out of the unit circle on the real line through the point -1, then period doubling bifurcation appears. The bifurcation associated with the appearance of $\mu_1 = -1$ is called a period-doubling bifurcation presented schematically in *Fig. 3*.
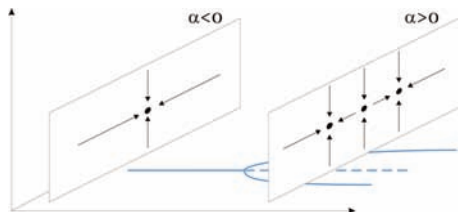

Fig. 3. An attracting fixed point loses stability α = 0 in a perioddoubling bifurcation. For α > 0 there is a saddle fixed point and a period-two attractor
Rys. 3. Przyciągający punkt stały traci stabilność dla α = 0 na drodze bifurkacji podwojenia okresu. Dla α > 0 istnieje siodłowy punkt stały oraz dwuokresowy atraktor

Periodic orbits of periods greater than one can appear or disappear because of saddle-node bifurcations (see *Fig. 4*), and can undergo period doubling bifurcations. This kind of behaviour exists in analysed DC-DC buck converter.
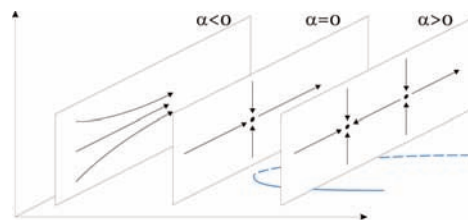

Fig. 4. A fixed point appears at parameter α = 0 in a saddle-node bifurcation. For α > 0 there is an attracting fixed point and a saddle fixed point
Fig. 4. Punkt stały pojawia się dla α = 0 na drodze bifurkacji siodło-węzeł. Dla α > 0 występuje przyciągający punkt stały oraz siodłowy punkt stały

At a period-doubling bifurcation from a period-k orbit, two branches of period-$2k$ points emanate from a path of period-k points. When the branches split off, the period-k points change stability.

Bifurcation can sometimes be catastrophic. For example, a converter may operate nicely when a certain parameter is kept below a certain threshold. Beyond this threshold, a chaotic attractor may suddenly take over, with its trajectory extended to a much wider voltage and current ranges causing damage to the devices. Thus, the study of bifurcation in an engineering system is relevant not only to its functionality but also to reliability and safety [6,8].

Computer simulation

Assuming the notation used previously, the parameters of the circuit are: R, C, and L, the resistance, the capacitance and the inductance of the circuit respectively. V_l and V_u, the lower and upper voltages of the ramp in feedback control loop and T its period, A is the gain of the amplifier and V_{ref} the reference voltage. V_{in} is the input voltage and established as the bifurcation parameter varied in interval [20 V, 35 V]. The buck converter is investigated using the following parameter values: $L = 20$ mH, $C = 47$ µF, $R = 22 < \Omega$, $A = 8.2$, $V_{ref} = 11.3$ V, $V_l = 3.8$ V, $V_u = 8.2$ V, ramp frequency $f = 2.5$ kHz.

Theoretical model simulation

One of the routes to chaos observed in studied DC-DC buck converter is by period doubling [5,6], which continues until there are no further stable states. At the beginning of simulation when input voltage is 20 V circuit exhibits periodic behaviour. During system bifurcation parameter changes, periodic state becomes unstable because of period doubling bifurcation.

In spectral analysis shown in *Fig. 5* it is observed as second frequency appearing at half the driving frequency. Further increase in input voltage results in splitting of two periods, giving quadrupling and finally chaos. In periodic system, only one harmonic peak occurs, associated with driving frequency. During bifurcation parameter variations the system changes its behaviour, more peaks occur, associated with harmonics and subharmonics of the system. This is called the period doubling cascade route to chaos. Because it is easy to vary, the input voltage V_{in} was chosen as the bifurcation parameter. The i_L, v_c and v_{co} were sampled at the start of every ramp cycle and plotted as the bifurcation diagram shown in *Fig. 6*. A period doubling route to chaos is visible. This process is repeated for every discrete value of the bifurcation parameter in the interval $V_{in} = [20,35]$ V.
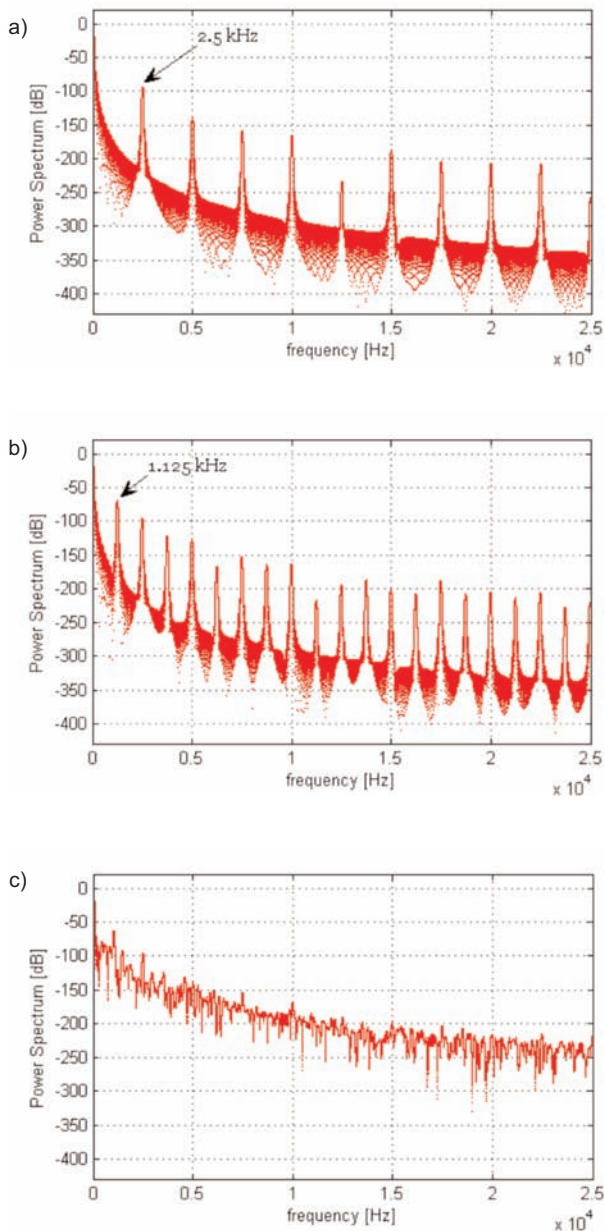
a)

b)

c)

Fig. 5. Spectral analysis of inductor current: a) 1*T*-periodic inductor current power spectrum density; b) 2*T*-periodic inductor current power spectrum density; c) chaotic inductor current power spectrum density
Rys. 5. Analiza spektralna prądu w cewce: a) widmo gęstości mocy 1-okresowego prądu w cewce; b) widmo gęstości mocy 2-okresowego prądu w cewce; c) widmo gęstości mocy chaotycznego przebiegu prądu w cewce

There was calculated that stable 1*T*-periodic limit cycle is found at the beginning of simulation and continued until some value near 24.5 V. Then, a first period doubling bifurcation occurs, and the stability of the 1*T*-periodic orbit is lost in favour of the 2*T*-periodic orbit. This 2*T*-periodic orbit also loses stability in a period doubling bifurcation near 31.15 V and 4*T*-periodic appears. Near the last period doubling bifurcation at approximately 32.4 V, there is a large chaotic behaviour.

Coexisting attractors are also able to detect in studied buck converter. When V_{in} is about 24 V, unstable chaotic orbits coexist with the periodic attractor, giving rise to a long transient chaotic behavior before the converter settles to the stable periodic orbit. A parallel branches of 6*T*-periodic orbit are detected in a neighbourhood of V_{in} = 30.000 V after sad-

dle-node bifurcation. This undergoes its own period-doubling cascade which ends in a six-piece chaotic attractor coexisting with the main 2*T*-periodic stable orbit.
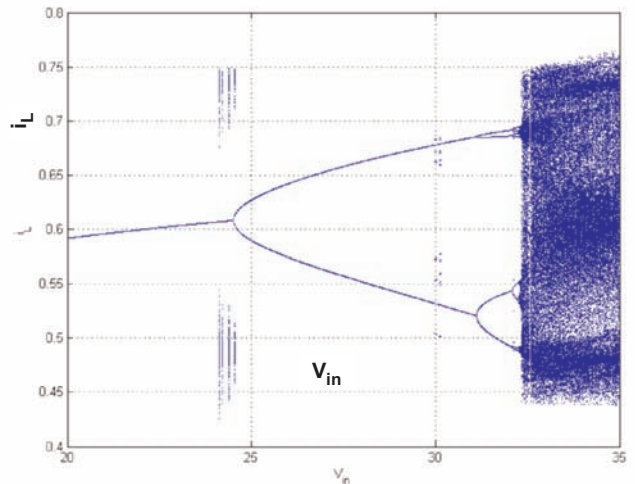

Fig. 6. Bifurcation diagram with V_{in} as the bifurcation parameter
Rys. 6. Diagram bifurkacyjny z V_{in} przyjętym jako parametr bifurkacyjny
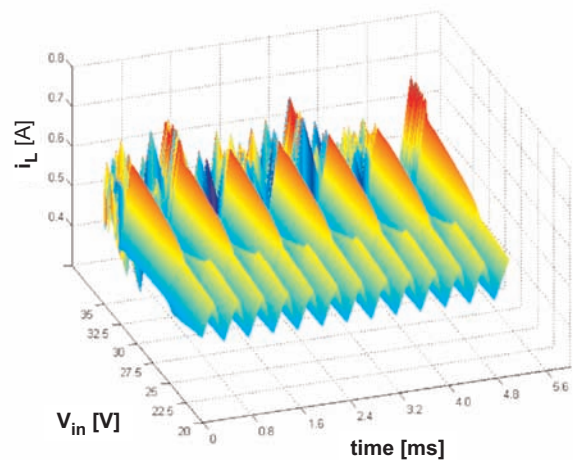

Fig. 7. Inductor current changes during period doubling cascade
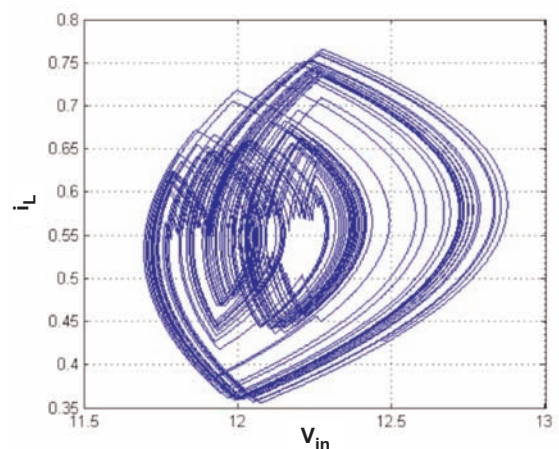Rys. 7. Prąd w cewce ulaga zmianie na drodze kaskady podwojenia okresu

Fig. 8. Strange attractor in DC-DC buck converter, inductor current against capacitor voltage
Rys. 8. Osobliwy atraktor zaobserwowany w układzie obniżającym napięcie, trajektorai fazowa przedstawiona na płaszczyźnie prąd-napięcie

Fig. 7 shows inductor current changes during bifurcation parameter variation. It is able to observe how the amplitude of current in inductor can increase from 1*T*-periodic waveform to chaotic. The maximum value was measured in chaotic operating mode. Strange attractor with non-integer dimension characteristic for chaotic behaviour is shown in *Fig. 8*.

The Poincaré section diagram come into being as a result of simulated waveforms sampling synchronised with the ramp voltage, one sample of the current and voltage variables at the beginning of the ramp. Then the representation in the state space of the points obtained with this procedure gives us the discrete evolution of the system. Period doubling route to chaos shown in Poincaré section is presented in *Fig. 9*.

Electrical circuit computer simulation

Before DC-DC buck converter physical realisation of the designed circuit was simulated in PSpice. The observation of effects characteristic for nonlinear dynamics like bifurcations and chaos in circuit with voltage feedback was carried out. In the designed circuit the power circuit uses a power metal-oxide-semiconductor field-effect transistor (MOSFET) IRF9640 and power standard recovery diode 1N4001. The DC input voltage was varied from 20 to 35 V as in mathematical model, control circuit was supplied from a voltage regulator LM7812. The ramp generator, based on a 555 timer, produces a sawtooth waveform. A bandwidht dual operational amplifier TL082 is used as a comparator and a difference amplifier.

The suggestion of practical controlled DC-DC buck converter is shown in *Fig. 10*. The circuit is closely related to these used in switch-mode power supplies. The coil current plotted against the capacitor voltage constitute strange attractor in chaotic operating mode which is shown in *Fig. 11*.

Practical verification

In order to verify the theoretical model an experimental buck converter was built. The main aim was to make its operation close to ideal piecewise linear model. Builded converter was very similar to simulated in PSpice. As distinct from PSpice simulation as the comparator is applied LM311 and as the difference amplifier is used a complementary metal-oxide-semiconductor (CMOS) operational amplifier LMC662.

A very good premise of chaotic behaviour presence is non-periodic attractor shown in *Fig. 12*. Likewise in computer simulation there is a possibility to observe chaotic phenomena in laboratory experiment. In *Fig. 13* there is a Poincaré section measured in a laboratory similar to obtained from the mathematical model computer simulation in *Fig. 9d*.

Comparison of resolutes

In order to verify obtained results from computer simulations and laboratory experiment there will be a comparison presented. The outcomes of mathematical model from Matlab in comparison with the results from PSpice simulated circuit and the physical laboratory experiment will be shown.
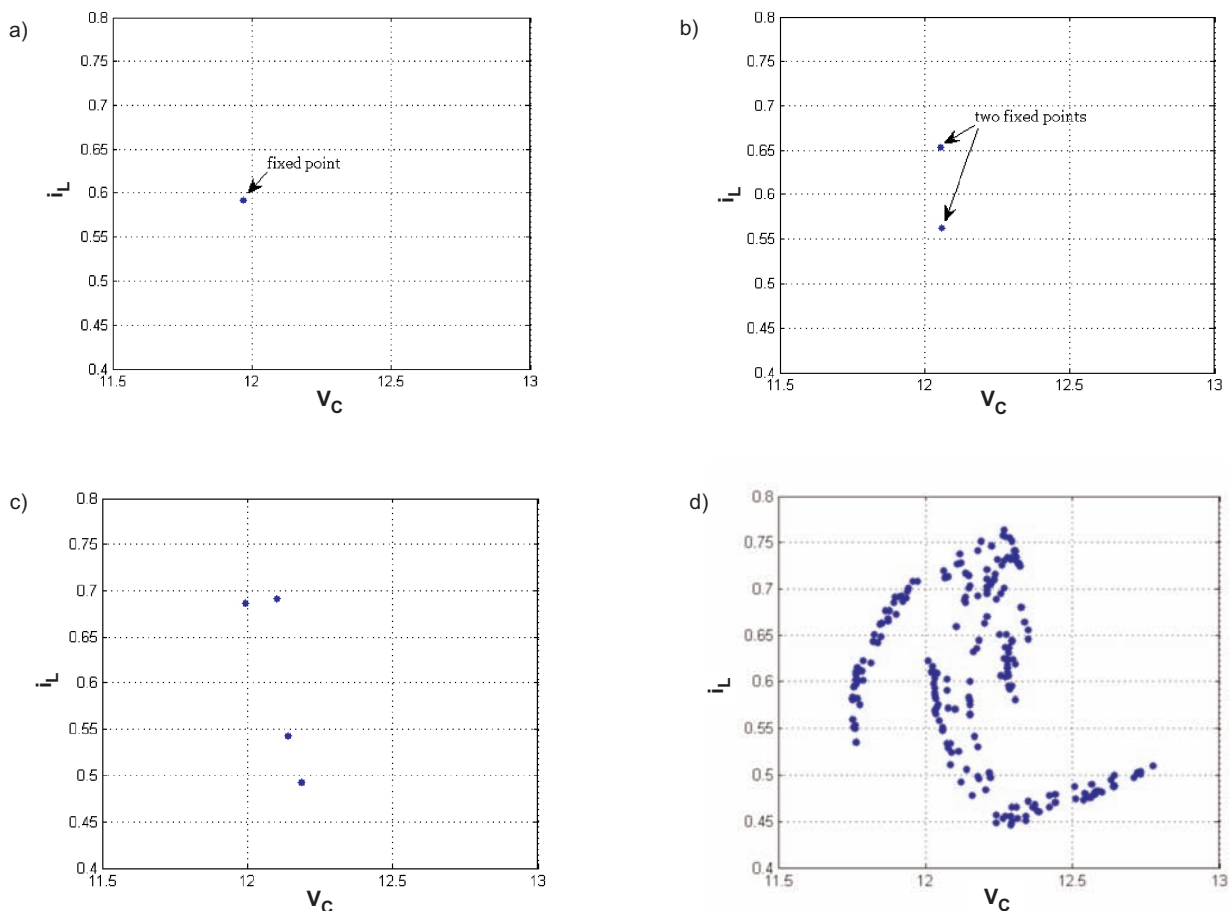

Fig. 9. Poincaré section diagram: a) 1*T*-periodic operation; b) 2*T*-periodic operation; c) 4*T*-periodic operation; d) chaotic operation
Rys. 9. Płaszczyzny Poincaré: a) 1-okresowa praca układu; b) 2-okresowa praca układu; c) 4-okresowa praca układu; d) praca chaotyczna
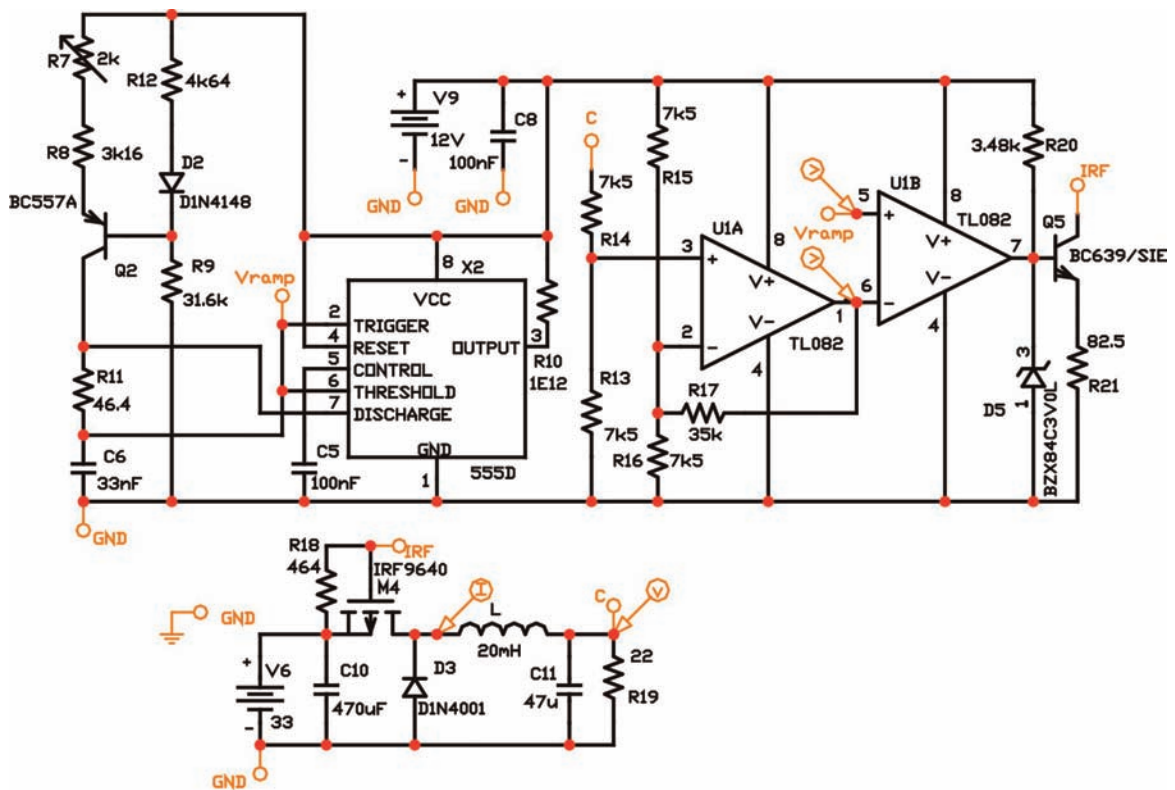
Fig. 10. Circuit diagram of the experimental buck converter
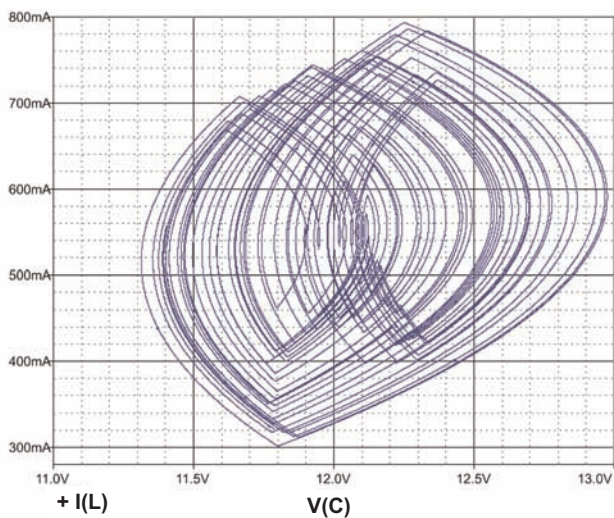**Rys. 10. Propozycja obwodu elektrycznego**

Fig. 11. Strange attractor in DC-DC buck converter obtained from PSpice
Rys. 11. Osobliwy atraktor w układzie obniżającym napięcie otrzymany z PSpice'a
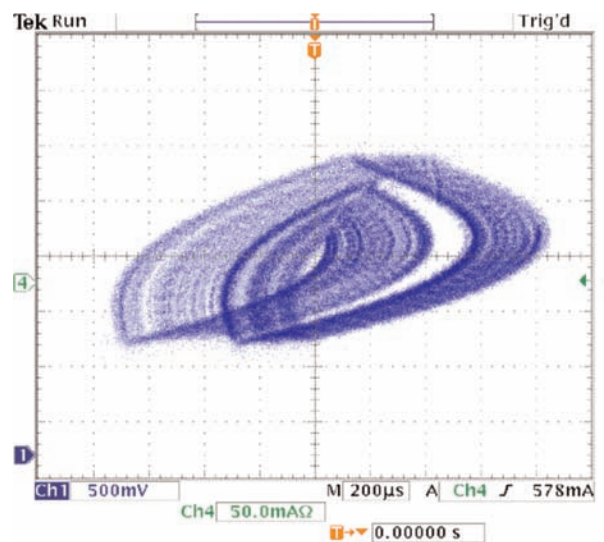

Fig. 12. Strange attractor in DC-DC buck converter measured in a laboratory
Rys. 12. Osobliwy atraktor występujący w układzie obniżającym napięcie zaobserwowany w laboratorium
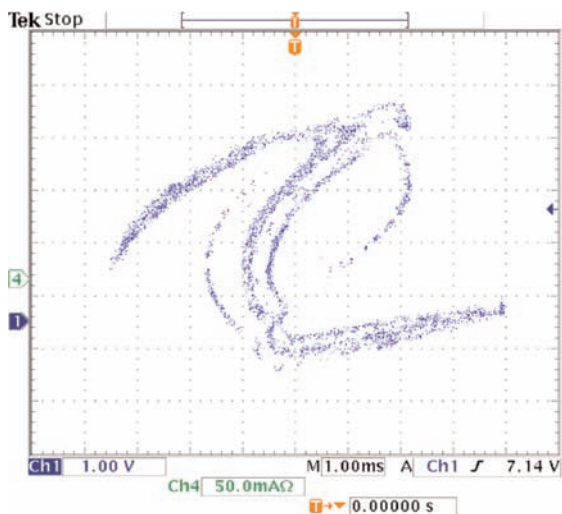
Fig. 13. Poincaré section characteristic for chaotic behaviour
Rys. 13. Płaszczyzna Poincaré charakterystyczna dla pracy chaotycznej układu

The results were obtained from three sources: mathematical model numerical calculations in Matlab; simulation using PSpice, with consideration of additional effects present in practical realisation; results from the laboratory experiment. The 4*T*-orbit was chosen for the sake of sufficient complexity and is presented in *Fig. 14*. Such a presentation enables clear observation of behaviour present in the system. This attractor appears after second flip bifurcation and is a premise of chaotic phenomena existed in the buck converter.

Conclusion

The main aim was to present nonlinear system analysing methods and its application in power electronics. The DC-DC second-order buck converter with the voltage control was taken as an example. The main objective was to build a converter which is able to work in chaotic operating mode basis of the mathematical model. Simultaneously there were shown analytical methods helpful in detecting, analysing and classifying this kind of nonlinear behaviour.

Additionally it was presented that nonlinear analysis describes analysed system more accurately and explains phenomenons such as subharmonics, bifurcations and chaos, which cannot be detected by using linear approach of analysis. It proves that it could be useful in circuits study, specially in a field where high reliability is essential, like in spacecraft power systems or terrestrial power systems.

The investigation was carried out in three different ways and the results were compared. There were considered three independent cases: the mathematical model simulated in Matlab, the circuit builded from components exist in reality and simulated in PSpice and the laboratory experiment. All of cases give satisfactory results and they were described in relevant sections. A very good agreement between theory and experiment was reached.

Fig. 14. 4*T*-periodic limit cycle comparison: a) 4*T*-periodic orbit from Matlab; b) 4*T*-periodic orbit from Pspice; c) 4*T*-periodic orbit obtained in a laboratory
Rys. 14. Porównanie 4-okresowego cyklu granicznego: a) orbita 4-okresowa wyznaczona w programie Matlab; b) orbita 4-okresowa z PSpice'a; c) orbita 4-okresowa otrzymana w laboratorium

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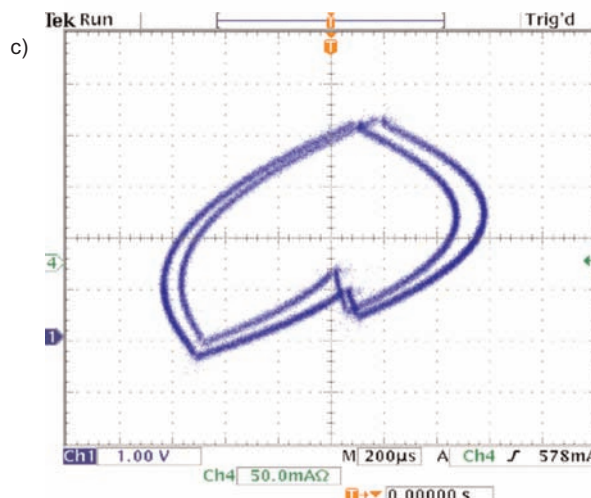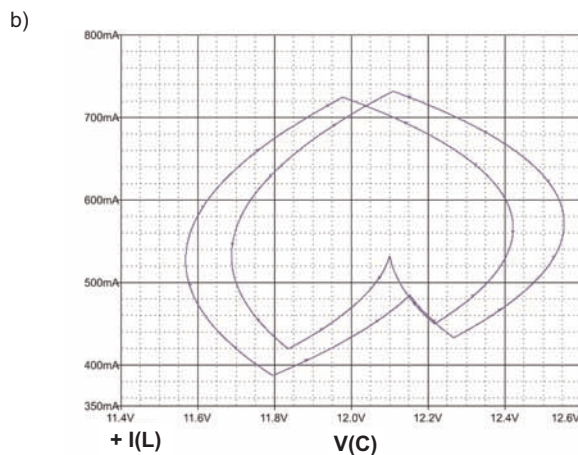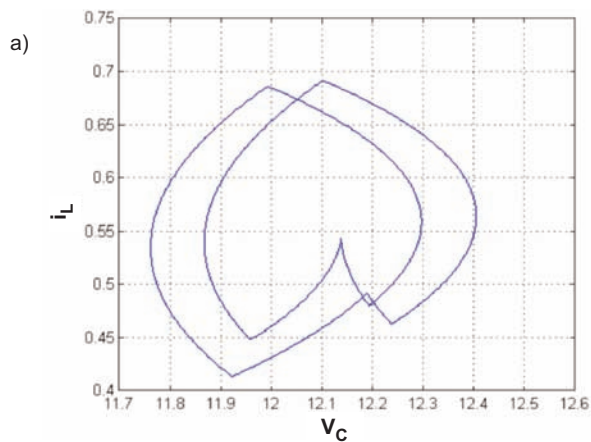[3] Kuznetsov Y. A.: Elements of Applied Bifurcation Theory. Springer-Verlag, 1998.
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